

Classical and Quantum Wormholes in a Flat Λ -Decaying Cosmology

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Abstract We study the classical and quantum wormholes for a flat *Euclidean* Friedmann-Robertson-Walker metric with a perfect fluid including an ordinary matter source plus a source playing the role of dark energy (decaying cosmological term). It is shown that classical wormholes exist for this model and the quantum version of such wormholes are consistent with the Hawking-Page conjecture for quantum wormholes as solutions of the Wheeler-DeWitt equation.

Keywords Classical-Quantum wormholes · Dark energy

1 Introduction

Classical wormholes are usually considered as Euclidean metrics that consist of two asymptotically flat regions connected by a narrow throat (handle). Wormholes have been studied mainly as instantons, namely solutions of the classical Euclidean field equations. In general, such wormholes can represent quantum tunneling between different topologies. They are possibly useful in understanding black hole evaporation [1]; in allowing nonlocal connections that could determine fundamental constants; and in vanishing the cosmological constant Λ [2–4]. They are even considered as an alternative to the Higgs mechanism [5]. Consequently, such solutions are worth finding.

Unfortunately, they exist for certain special kinds of matter [6–10]. For example, they exist for an imaginary minimally coupled scalar field [11], but do not exist for pure gravity. Due to limited known classical wormhole solutions, Hawking and Page advocated a different approach in which wormholes were regarded, not as solutions of the classical Euclidean

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field equations, but as solutions of the quantum mechanical Wheeler-DeWitt equation [11]. These wave functions have to obey certain boundary conditions in order that they represent wormholes. The boundary conditions seem to be that: (i) the wave function is exponentially damped for large tree geometries, namely 3-dimensional space-like hypersurfaces with large scale factor; (ii) the wave function is regular in some suitable way when the tree-geometry collapses to zero.

The first condition expresses the fact that spacetime should be Euclidean when tree geometries become infinite. The second condition should reflect the fact that spacetime is nonsingular when tree geometries degenerates, namely the wave function should not oscillate an infinite number of time.

Therefore, in general, an open and interesting problem is whether classical and quantum wormholes can occur for fairly general matter sources. To the author's knowledge, the dark energy as a Λ -decaying source has not been received much attention, in this regard. Although there are few works attempting to study such a universe (in a different formulation), using quantum effective action [12–15], we shall consider this model and seek for the classical and quantum wormhole solutions, explicitly. In fact, there are theoretical and observational motivations for considering models in which Λ decays. For example, such behaviors as $\Lambda \sim t^{-2}$ [17] or $\Lambda \sim a^{-m}$ [18, 19] have already been reported. For $0 \leq m < 3$ [18, 19], the effect of decaying cosmological term on the cosmic microwave background anisotropy is studied and the angular power spectrum for different values of m and density parameter Ω_{m0} is computed. Models with $\Omega_{m0} \geq 0.2$ and $m \geq 1.6$ are shown to be in good agreement with data. For $m = 2$ [20], it is shown that in the early universe Λ could be several tens of orders bigger than its present value, but not big enough disturbing the physics in the radiation-dominant epoch in the standard cosmology. In the matter-dominant epoch such a varying Λ shifts the three space curvature parameter k by a constant which changes the standard cosmology predictions reconciling observations with the inflationary scenario. Such a vanishing cosmological term also leads to present creation of matter with a rate comparable to that in the steady-state cosmology [20].

2 Classical Wormholes

The reason why classical wormholes may exist is related to the implication of a theorem of Cheeger and Glomol [16] which states that a *necessary* condition for a wormhole to exist is that the eigenvalues of the Ricci tensor be negative *somewhere* on the manifold. This is necessary but not *sufficient* condition for their existence. For example, the energy-momentum tensors of an axion field and of a conformal scalar field are such that, when coupled to gravity, the Ricci tensor has negative eigenvalues. However, for a minimally coupled scalar field with the lagrangian density

$$\mathcal{L} = \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi,$$

and the energy-momentum tensor

$$T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla_\lambda \phi \nabla^\lambda \phi,$$

the Einstein equations read

$$\mathcal{R}_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi,$$

which shows $\mathcal{R}_{\mu\nu}$ can never be negative.

Taking this into account, we consider an Euclidean Friedmann-Robertson-Walker (FRW) metric

$$ds^2 = dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right], \tag{1}$$

where $a(t)$ is the scale factor and $k = 0, +1$ and -1 account for flat, closed and open universes, respectively. This metric evolves according to the Einstein equation¹

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = T_{\mu\nu}, \tag{2}$$

where we take the energy momentum tensor $T_{\mu\nu}$ to be perfect fluid

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}. \tag{3}$$

The time-time and space-space components of the Einstein equation (2) leads respectively to

$$\frac{\dot{a}^2}{a^2} = \frac{k}{a^2} - \frac{\rho}{3}, \tag{4}$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} = p, \tag{5}$$

where (4) is the Euclidean Friedmann equation in which the derivative is with respect to the Euclidean time. There is also a conservation equation $\nabla_\mu T^{\mu\nu} = 0$ whose time component gives the fluid equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0. \tag{6}$$

Wormholes are typically described by the Euclidean Friedmann equation (4) with a scale factor dependent density, typically as $\rho \sim a^{-n}$. We now assume a *flat* FRW universe with a perfect fluid source combined of ordinary matter with the equation of state $p = (\gamma - 1)\rho$ and a source evolving with the scale factor

$$\rho = \rho_m + \rho_v = \left(\frac{\rho_0}{a^{3\gamma}} - \frac{\Lambda_0}{a^2} \right). \tag{7}$$

The first term is the ordinary matter density and the second term is a density playing the role of (negative) dark energy—a decaying cosmological term.

To obtain the necessary condition for the existence of wormholes we evaluate the eigenvalues of the Ricci tensor²

$$\mathcal{R}_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T, \tag{8}$$

which together with the perfect fluid (3) and the equation of state (11) (see below) leads, for example, to

$$\mathcal{R}_{00} = \frac{\rho_0}{a^{3\gamma}} \left[1 - \frac{\gamma}{2} \right] - \frac{4}{3} \frac{\Lambda_0}{a^2}. \tag{9}$$

¹We have used the units where $8\pi G = c = 1$.

²Note that since both the Robertson-Walker metric $g_{\mu\nu}$ and the energy-momentum tensor $T_{\mu\nu}$ are diagonal, then Ricci tensor is also diagonal and its diagonal components become its eigenvalues.

It is seen that \mathcal{R}_{00} can be negative at large a for any $\gamma > 2/3$. However, it does not guarantee that wormholes certainly do exist for this range of γ . It is therefore, necessary to check out the *sufficient* condition.

Substitution for ρ from (7) and $k = 0$ into (4) leads to³

$$\frac{\dot{a}^2}{a^2} = \frac{1}{3} \left(\frac{\Lambda_0}{a^2} - \frac{\rho_0}{a^{3\gamma}} \right). \tag{10}$$

In order to have an asymptotically Euclidean wormhole \dot{a}^2 must remains positive at large a . This wormhole then represents two separate asymptotically Euclidean regions joined together by a throat with the finite size a_0 at which $\dot{a} = 0$. Since according to *strong energy condition* $\rho_0 > 0$, then we can not have this wormhole for $\Lambda_0 < 0$ because \dot{a}^2 becomes negative. Therefore, classical Euclidean wormholes are possible for $\Lambda_0 > 0$, namely if the dark energy has a negative energy density $\rho_v = -\frac{\Lambda_0}{a^2} < 0$. On the other hand, even for $\Lambda_0 > 0$ we need $\gamma > \frac{2}{3}$ to have a positive \dot{a}^2 at large a .

To determine the equation of state, we substitute for ρ from (7) in the conservation equation (6) and obtain the following equation of state

$$\begin{aligned} p &= p_m + p_v \\ &= \rho_m(\gamma - 1) - \frac{1}{3}\rho_v, \end{aligned} \tag{11}$$

where the first term describes the standard equation of state for ordinary matter and the second term accounts for equation of state for the dark energy.

In conclusion, considering the *necessary* and *sufficient* conditions as $\gamma > \frac{2}{3}$, one finds that classical Euclidean wormholes are possible for the matter source (7) if and only if $\gamma > \frac{2}{3}$, provided the dark energy has a negative energy density. In the Lorentzian sector of the model, it is easy to show that $\gamma > \frac{2}{3}$ leads the strong energy condition to hold for the total pressure p and total density ρ .

3 Quantum Wormholes

As a matter of fact, the number of known classical wormholes is so limited. It casts doubt on whether wormholes are important, only in the very restricted class of theories, in which the matter content allows wormhole instantons? To resolve this problem, Hawking and Page advocated a different approach and considered that solutions of the Wheeler-DeWitt equation could more generally represent the wormholes [11]. They realized that for the mini superspace models one may consider metrics of the Euclidean Friedmann form

$$ds^2 = N^2(t)dt^2 + a^2(t)d\Omega_3^2. \tag{12}$$

If N is imaginary, this is the Lorentzian metric, and if N is real, it is the metric of an Euclidean wormhole. However, solutions of the Wheeler-DeWitt equation are independent of the lapse function N and t . So, they can be interpreted either as Friedmann universe, or as wormholes according to the appropriate boundary conditions. The boundary conditions for wormholes seems to be that the wave functions should decay exponentially for large scale

³Recent observations strongly indicate that the universe is almost flat, namely $k = 0$.

factor a , so as to represent Euclidean space, and that they be regular in some suitable way as $a \rightarrow 0$, so that no singularities are present.

By defining $R = \sqrt{\frac{3}{\Lambda_0}} a$ we obtain the rescaled Friedmann equation

$$\frac{\dot{R}^2}{R^2} = \frac{1}{R^2} - \frac{\alpha_0}{R^{3\gamma}}, \tag{13}$$

with the constant

$$\alpha_0 = \frac{\rho_0}{\left(\frac{3}{\Lambda_0}\right)^{3\gamma/2}}. \tag{14}$$

The quantum mechanical version of this equation is given by [22]

$$\left(R^2 \frac{d^2}{dR^2} + qR \frac{d}{dR} + \alpha_0 R^{6-3\gamma} - R^4 \right) \Psi(R) = 0, \tag{15}$$

where q represents part of the factor ordering ambiguities. We set $q = 0$ and study the potential to get some idea as to when a Euclidean domain occurs at large R by considering the sign of the potential

$$U(R) = \alpha_0 R^{4-3\gamma} - R^2, \tag{16}$$

in the equation

$$\left[\frac{d^2}{dR^2} + U(R) \right] \Psi(R) = 0. \tag{17}$$

For positive potential $U(R) > 0$, oscillating solutions occur which represent Lorentzian metrics. On the contrary, for negative potential $U(R) < 0$, wormhole solutions can occur which are asymptotically Euclidean at large R . The potential is negative for $\gamma > 2/3$ in the case of positive energy density ρ_0 . Therefore, wormholes obeying the Hawking-Page boundary condition at large R , occur when $\gamma > 2/3$ is valid for the source (7). The presence of any matter source with $\gamma < 2/3$ will eventually dominate for large R and prevent the asymptotically Euclidean wormholes to occur.

Asymptotically Euclidean property of the wave function is not sufficient to make it a wormhole. It also requires regularity for small R . In order to realize this, we can ignore R^4 term in (15) as $R \rightarrow 0$, when $\gamma > 2/3$. In this case, the Wheeler-DeWitt equation (15) (for $\gamma \neq 2$) simplifies to a Bessel differential equation with solution

$$\Psi(R) \simeq R^{(1-q)/2} \left[c_1 J_\nu \left(\frac{2\sqrt{\alpha_0}}{3(2-\gamma)} R^{3-3\gamma/2} \right) + c_2 Y_\nu \left(\frac{2\sqrt{\alpha_0}}{3(2-\gamma)} R^{3-3\gamma/2} \right) \right], \tag{18}$$

where use has been made of $\nu \equiv (1-q)/3(2-\gamma)$. The wormhole boundary condition at $R \rightarrow 0$ is satisfied for the Bessel function of the J kind.

In the particular case $\gamma = 2$, the solution of (15) with $q = 1$ is a linear combination of Bessel functions $J_{\pm i\sqrt{\alpha_0}/2(ia^2/2)}$ which oscillates an infinite number of times at $R \rightarrow 0$ and therefore can not satisfy the required regularity condition for a quantum wormhole.

On the other hand, in the case of $\gamma = 4/3$ which represents radiation (or equivalently, that of a conformally coupled scalar field) dominated FRW ansatz, (15) for $q = 0$ is written as

$$\left(\frac{d^2}{dR^2} + \alpha_0 - R^2 \right) \Psi(R) = 0, \tag{19}$$

Fig. 1 A quantum wormhole solution for the case $\gamma = 4/3$ with $c_3 = 0$ and $\alpha_0 = 35$

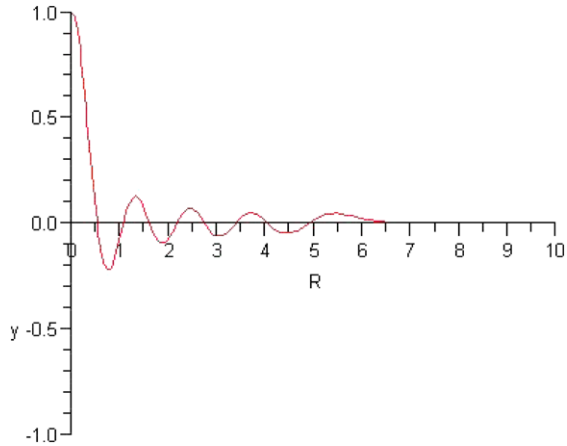
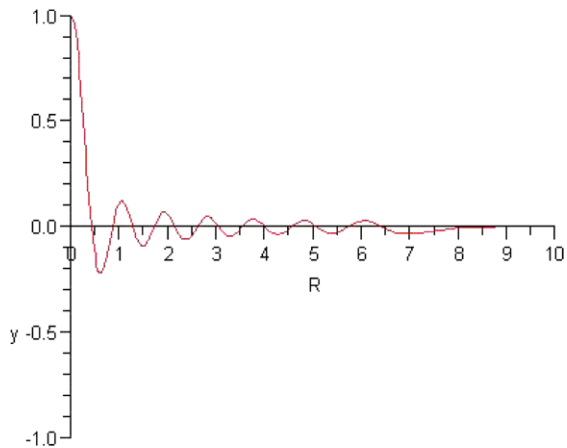


Fig. 2 A quantum wormhole solution for the case $\gamma = 4/3$ with $c_3 = 0$ and $\alpha_0 = 55$



which is in the form of a parabolic equation with solution in terms of confluent hypergeometric functions [23]

$$\Psi(R) \simeq \exp(-R^2/2) \left[c_{31} F_1 \left(\frac{1}{4}(1 - \alpha_0); 1/2; R^2 \right) + c_{41} F_1 \left(\frac{1}{4}(3 - \alpha_0); 3/2; R^2 \right) \right]. \tag{20}$$

For example, for $c_3 = 0$ with $\alpha_0 = (35, 55)$, and $c_4 = 0$ with $\alpha_0 = (25, 37)$, we obtain regular oscillations at $R \rightarrow 0$, and Euclidean regimes for large R , see Figs. 1, 2, 3, 4.

Therefore, Hawking-Page boundary conditions are satisfied for some special values of α_0 and so we have a spectrum of wormholes. Considering (14), it turns out that for a given equation of state with $\gamma > 2/3$, the existence of quantum wormholes depends on the special values of ρ_0 and Λ_0 . In other words, the spectrum of wormholes depends on the spectrum of ρ_0 and Λ_0 . We notice that there is no such constraint on these values for the occurrence of classical wormholes.

Fig. 3 A quantum wormhole solution for the case $\gamma = 4/3$ with $c_4 = 0$ and $\alpha_0 = 25$

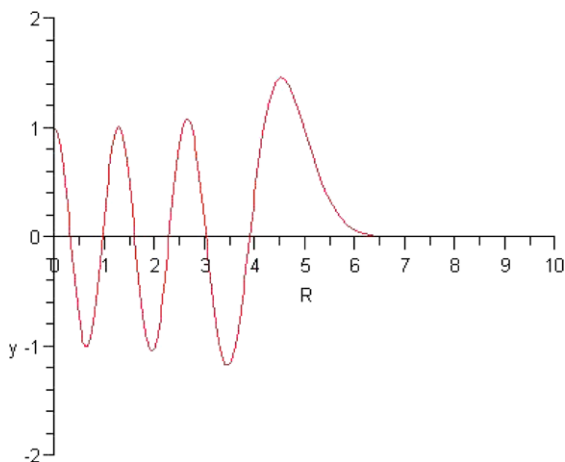
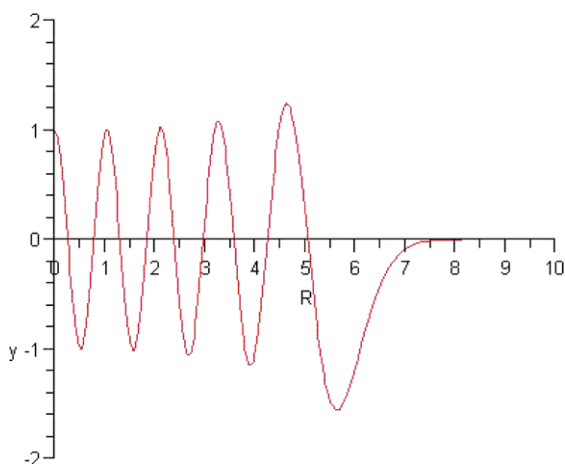


Fig. 4 A quantum wormhole solution for the case $\gamma = 4/3$ with $c_4 = 0$ and $\alpha_0 = 37$



4 Conclusion

Any wormhole like solution is of particular importance from macroscopic and microscopic points of view. In particular, very small wormholes are studied as instantons, namely the saddle points in the Euclidean path integrals. So, one can use them to give a semi-classical treatment in the dilute wormhole approximation where the interaction between the large scale ends of wormholes is neglected. On the other hand, since the black holes evaporate in theories with any reasonable matter content, then any new wormhole like solution may provide a new contribution for black hole evaporation. In the same way, any new wormhole like solution is supposed to play its own important role in vanishing the cosmological constant. Taking into account the importance of new wormhole like solutions, as discussed above, we have studied the classical and quantum wormhole solutions for a flat Euclidean Friedmann-Robertson-Walker metric coupled to a perfect fluid combined of an ordinary matter source and a source playing the role of dark energy (a decaying cosmological term). We have shown that classical and quantum wormholes exist for this matter source for which the strong energy condition could be hold in the Lorentzian sector of the model. An spectrum of quantum

wormholes has been obtained, for each of which there is a specific relation (14) between the characteristics of matter and dark energy densities. These new wormhole like solutions, among others obtained so far, may be used as the saddle points in the Euclidean path integrals to give a semi-classical treatment in the dilute wormhole approximation. They may also contribute to the black hole evaporation as well as vanishing cosmological constant.

It is important to note that in this model one can not expect wormholes without the decaying cosmological term: If $\Lambda_0 = 0$, although the eigenvalues of Ricci tensor, according to (9), may become negative for $\gamma > 2$ and necessary condition is satisfied, but according to (10), \dot{a}^2 can not become positive, so the sufficient condition is not satisfied.

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References

1. Hawking, S.W.: Phys. Rev. D. **37**, 904 (1988)
2. Coleman, S.: Nucl. Phys. B **307**, 864 (1988)
3. Coleman, S.: Nucl. Phys. B **310**, 643 (1988)
4. Klebanov, I., Susskind, L., Banks, T.: Nucl. Phys. B **317**, 665 (1989)
5. Mignemi, S., Moss, I.: Phys. Rev. D **48**, 3725 (1993)
6. Giddings, S., Strominger, A.: Nucl. Phys. B **306**, 890 (1988)
7. Lee, K.: Phys. Rev. Lett. **61**, 263 (1988)
8. Hosoya, A., Ogura, W.: Phys. Lett. B **225**, 117 (1989)
9. Gupta, A.K., Hughes, J., Preskill, J., Wise, M.B.: Nucl. Phys. B **333**, 195 (1990)
10. Coule, D.H., Maeda, K.: Class. Quantum Gravity **7**, 955 (1990)
11. Hawking, S., Page, D.N.: Phys. Rev. D. **42**, 2655 (1990)
12. Nojiri, S., Obregon, O., Odintsov, S.D., Osetrin, K.E.: Phys. Lett. B **449**, 173 (1999)
13. Nojiri, S., Obregon, O., Odintsov, S.D., Osetrin, K.E.: Phys. Lett. B **458**, 19 (1999)
14. Carlini, A., Mijic, M.: SISSA preprint, **91A** (1990)
15. Carlini, A., Coule, D.H., Solomons, D.M.: Mod. Phys. Lett. A **11**, 1453 (1996)
16. Cheeger, J., Gromoll, D.: Ann. Math. **96**, 413 (1972)
17. Bertolami, O.: Nuovo Cim. B **93**, 36 (1986)
18. Silveira, V., Waga, I.: Phys. Rev. D **56**, 4625 (1997)
19. Silveira, V., Waga, I.: Phys. Rev. D **50**, 4890 (1994)
20. Chen, W., Wu, Y.-S.: Phys. Rev. D **41**, 695 (1990)
21. Zhuk, A.: Phys. Lett. A **176**, 176 (1993)
22. Halliwell, J.J.: In: Quantum Cosmology and Baby Universes. World Scientific, Singapore (1991)
23. Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions. Dover, New York (1965)